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Parallel spinors on flat manifolds

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Abstract

Let $\mathfrak{p}(M)$ be the dimension of the vector space of parallel spinors on a closed spin manifold M. We prove that every finite group G is the holonomy group of a closed flat spin manifold $M(G)$ such that $\mathfrak{p}(M(G)) > 0$. If the holonomy group $\mathrm{Hol}(M)$ of M is cyclic, then we give an explicit formula for $\mathfrak{p}(M)$ another than that given in [R.J. Miatello, R.A. Podesta, The spectrum of twisted Dirac operators on compact flat manifolds, Trans. Am. Math. Soc., in press]. We answer the question when $\mathfrak{p}(M) > 0$ if $\mathrm{Hol}(M)$ is a cyclic group of prime order or $\dim M \leq 4$.

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1. Introduction

In this paper, we consider the question of the existence of parallel spinors on a closed flat manifold M. Parallel spinors on M coincide with harmonic ones. Our first result is a variant of the Auslander–Kuranishi theorem showing that every finite group is the holonomy group

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of a closed flat spin manifold admitting nontrivial parallel spinors. The other ones deal with the dimensions of vector spaces of parallel spinors on flat manifolds with cyclic holonomy groups and flat 4-manifolds. Our results and Theorem 2.6 in [11] are complementary to that of [12] and [15], where the question of the existence of parallel spinors was considered for irreducible nonflat Riemannian manifolds. An analogous problem for harmonic spinors has been investigated in many papers (see e.g. [2,8–10] and the references given there).

To explain the formulations of the results we need some definitions. An n-dimensional closed flat manifold M is the orbit space of the Euclidean space $\mathbb{R}^n$ by a properly discontinuous and free action of a discrete subgroup Γ of the isometry group $I(\mathbb{R}^n)$ of $\mathbb{R}^n$. The holonomy homomorphism $\varphi : \Gamma \to \mathrm{O}(n)$ carries $\gamma \in \Gamma$ onto the linear part of γ. The manifold M is orientable if and only if $\operatorname{im}\varphi \subset \mathrm{SO}(n)$. The compactness of M implies that $\ker\varphi \cong \mathbb{Z}^n$. The finite group $\varphi(\Gamma)$ (denoted by $\mathrm{Hol}(M)$) is the holonomy group of M (cf. [4, p. 51], [18, ch. 3, Lemma 3.4.4]). If M is a spin manifold, then every spin structure $\mathfrak{s}$ on M can be identified with a lift $\widehat{\varphi} : \Gamma \to \mathrm{Spin}(n)$ of $\varphi : \Gamma \to \mathrm{SO}(n)$. Let $\mathfrak{p}(M, \mathfrak{s})$ denote the dimension of the vector space of parallel spinors on M with a spin structure $\mathfrak{s}$. Our first two results are the following.

Theorem 1.1. *For every finite group G there are a closed flat manifold $M(G)$ and a spin structure $\mathfrak{s}$ on $M(G)$ such that* $\mathrm{Hol}(M(G)) \cong G$ *and* $\mathfrak{p}(M(G), \mathfrak{s}) > 0$.

Theorem 1.2. *Let M be a closed flat 4-manifold. The following conditions are equivalent*:

(a) *there is a spin structure $\mathfrak{s}$ on M such that* $\mathfrak{p}(M, \mathfrak{s}) > 0$,
(b) *M is a torus.*

To formulate the next results we need more definitions. We say that a spin-structure $\widehat{\varphi}$ is *admissible* if $\widehat{\varphi} = \iota_a \circ \varphi$ for some homomorphism $\iota_a : \varphi(\Gamma) \to \mathrm{Spin}(n)$. It is known that only admissible spin structures admit parallel spinors (cf. Lemma 2.1 below). Now assume that $M = \mathbb{R}^n/\Gamma$ is a closed, orientable, flat manifold with holonomy group $\varphi(\Gamma)$ isomorphic to $\mathbb{Z}_r$. Fix a generator A of $\varphi(\Gamma)$ and an orthonormal basis $e_1, \ldots, e_n$ in $\mathbb{R}^n$ such that $A|_{\mathrm{Span}[e_{2j-1}, e_{2j}]}$, $j = 1, \ldots, l$, is a rotation by $\widehat{\beta}_j$ and $A(e_j) = e_j$ for $j > 2l$. Let $\lambda : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ be the projection (cf. Section 2) and $k = [n/2]$. Take $\beta_j = \frac{1}{2}\widehat{\beta}_j$ and $C_j \in \{0, 1, \ldots, r-1\}$ such that $\beta_j = C_j \frac{\pi}{r}$. Setting $\widehat{\beta}_j = C_j = 0$ for $j > 2l$, we can assume that $l = k$. Consider

$$\rho_j = \cos\, \beta_j + e_{2j-1}e_{2j}\, \sin\, \beta_j$$

and $\alpha = \prod_{j=1}^{k} \rho_j$. It is easily seen that $\alpha \in \mathrm{Spin}(n)$ and $\lambda(\alpha) = A$. If $\widehat{\alpha} \in \{-\alpha, \alpha\}$ and $\widehat{\alpha}^r = 1$, then the formula $\iota_a(A) = \widehat{\alpha}$ defines an admissible spin structure on M that will be also denoted by $\widehat{\alpha}$. Let $\mu_I = \sum_{j=1}^{k} C_j$,

$$\alpha_+ = \begin{cases} \alpha & \text{if } \mu_I \text{ is even} \\ -\alpha & \text{if } \mu_I \text{ is odd,} \end{cases}$$

and $\alpha_- = -\alpha_+$. For every $\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \{-1, 1\}^k$ consider

$$\mu_\epsilon = \sum_{j=1}^{k} C_j \epsilon_j.$$

A Riemannian manifold with the holonomy group G will be called a *hol*(G)-*manifold*.

Theorem 1.3. *Let M be a closed* $\mathrm{hol}(\mathbb{Z}_r)$-*manifold and let n, k, μ_ϵ be as above. Assume that r is odd. Then α_+ is the unique admissible spin structure on M,*

$$\mathfrak{p}(M, \alpha_+) = \#\{\epsilon \in \{-1, 1\}^k : \mu_\epsilon \equiv 0 \bmod (2r)\} \quad if \;\; \mu_I \;\; is\ even,$$

and

$$\mathfrak{p}(M, \alpha_+) = \#\{\epsilon \in \{-1, 1\}^k : \mu_\epsilon \equiv r \bmod (2r)\} \quad if \;\; \mu_I \;\; is\ odd.$$

Theorem 1.4. *Let M be a closed* $\mathrm{hol}(\mathbb{Z}_r)$-*manifold and let n, k, μ_ϵ be as above. Assume that r is even.*

(a) *If μ_I is odd, then $\mathfrak{p}(M, \mathfrak{s}) = 0$ for every flat spin structure $\mathfrak{s}$ on M.*
(b) *If μ_I is even, then α and $-\alpha$ determine two admissible spin structures on M,*

$$\mathfrak{p}(M, \alpha) = \#\{\epsilon \in \{-1, 1\}^k : \mu_\epsilon \equiv 0 \bmod (2r)\},$$

and

$$\mathfrak{p}(M, -\alpha) = \#\{\epsilon \in \{-1, 1\}^k : \mu_\epsilon \equiv r \bmod (2r)\}.$$

Corollary 1.5. *If $\mathfrak{s}$ is a spin structure on a closed* $\mathrm{hol}(\mathbb{Z}_r)$-*manifold M, then $\mathfrak{p}(M, \mathfrak{s})$ is even.*

A general formula for $\mathfrak{p}(M, \widehat{\varphi})$ can be found in [11]. It shows that

$$\mathfrak{p}(M, \widehat{\varphi}) = \sum_{g \in \widehat{\varphi}(\Gamma)} \chi_{\widehat{\varphi}}(g),$$

where $\chi_{\widehat{\varphi}}$ is the character of the representation of $\widehat{\varphi}(\Gamma)$ determined by the action of $\widehat{\varphi}(\Gamma)$ on the irreducible $\mathrm{Cliff}_{\mathbb{C}}(n)$-module Σ_n. Our description of $\mathfrak{p}(M, \widehat{\varphi})$ is more convenient for us. If r is a prime number, then Theorems 1.3 and 1.4 together with the Diedrichsen–Reiner description of $\mathbb{Z}[\mathbb{Z}_r]$-lattices imply the following.

Theorem 1.6. *Let p be an odd prime number, let $M = \mathbb{R}^n / \Gamma$ be a closed* $\mathrm{hol}(\mathbb{Z}_p)$-*manifold, and let $l(M) = \frac{1}{p-1}[n - \dim(\mathbb{R}^n)^{\varphi(\Gamma)}]$. The following conditions are equivalent*:

(*a*) $\mathfrak{p}(M, \alpha_+) > 0$,
(*b*) $p > 5$ *or* $l(M) \geq 2$.

Theorem 1.7. *Let* $M = \mathbb{R}^n/\Gamma$ *be a closed* $\mathrm{hol}(\mathbb{Z}_2)$*-manifold and let* $l(M) = \frac{1}{2}[n - \dim(\mathbb{R}^n)^{\varphi(\Gamma)}]$. *The following conditions are equivalent*:

(*a*) $\mathfrak{p}(M, \alpha) > 0$,
(*b*) $\mathfrak{p}(M, -\alpha) > 0$,
(*c*) $l(M)$ *is even*.

If the order r of $\mathrm{Hol}(M)$ is not prime, then the question concerning the existence of parallel spinors is more difficult, because the integral representations of $\mathbb{Z}_r$ are more complicated. The motivations to study parallel spinors on flat manifolds with cyclic holonomy groups are the following. The case of cyclic holonomy is one of the simplest and one of the most important cases. It is the starting point of the investigation of parallel spinors on closed flat manifolds (cf. Section 3).

2. Holonomy groups and parallel spinors

The aim of this section is to prove Theorem 1.1. First we state some properties of parallel spinors on flat manifolds. For the details we refer to [8] and [13]. Let $\mathrm{Cliff}(n)$ be the Clifford algebra determined by the scalar product in $\mathbb{R}^n$, $\mathrm{Cliff}_{\mathbb{C}}(n) = \mathrm{Cliff}(n) \otimes \mathbb{C}$, and let $*$ be the anti-involution on $\mathrm{Cliff}(n)$ given on the generators $e_{i_1}, \ldots, e_{i_m}$ by the formula $(e_{i_1}, \ldots, e_{i_m})^* = e_{i_m}, \ldots, e_{i_1}$. The group $\mathrm{Spin}(n)$ consists of the products of even number of copies of the elements of the unit sphere in $\mathbb{R}^n$ and the standard covering map $\lambda : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ carries $y \in \mathrm{Spin}(n)$ onto $\lambda(y) : \mathbb{R}^n \ni x \to yxy^* \in \mathbb{R}^n$. The kernel of λ is equal to $\{-1, 1\}$. The irreducible complex $\mathrm{Cliff}_{\mathbb{C}}(n)$-module Σ_n can be described as follows (cf. [10, Section 1.3]). Consider

$$g_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

Let $k = [n/2]$, $\Sigma_n = \underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{k\,\text{times}}$, and $\alpha(j) = \begin{cases} 1 & \text{if } j \text{ is odd} \\ 2 & \text{if } j \text{ is even} \end{cases}$. Take an element $u = u_1 \otimes \cdots \otimes u_k$ of Σ_n and an orthonormal basis $e_1, \ldots, e_n$ in $\mathbb{R}^n$. For $j \leq 2k$ set

$$e_j u = (I \otimes \cdots \otimes I \otimes g_{\alpha(j)} \otimes \underbrace{T \otimes \cdots \otimes T}_{\left[\frac{j-1}{2}\right]\,\text{times}})(u).$$

If $n = 2k + 1$, then

$$e_n u = i(T \otimes \cdots \otimes T)u.$$

The vector space of parallel spinors on M can be identified with $\Sigma_n^{\operatorname{im}\widehat{\varphi}}$ (see e.g. [15, Section 4]). It is easy to check the following.

Lemma 2.1. *If* $\mathfrak{p}(M) > 0$, *then* $-1 \notin \operatorname{im}\widehat{\varphi}$.

Lemma 2.1 is a particular case of a more general observation ([14, Lemma 3.1], [15]). Before proving Theorem 1.1, we describe a useful construction. It has been used earlier in [6,7] and in the other papers. Let M_0 be a closed flat manifold of dimension n_0. Consider the Bieberbach exact sequence

$$1 \to \Delta_0 \to \Gamma_0 \xrightarrow{\varphi_0} G_0 \to 1,$$

where φ_0 is the holonomy homomorphism of M_0 and $\Delta_0 = \ker\varphi_0 \cong \mathbb{Z}^{n_0}$. This sequence is described by the action of G_0 on Δ_0 and the cohomology class $\nu_0 \in H^2(G_0, \Delta_0)$ corresponding to our extension. Fix $r \in \mathbb{N}$. Take

$$\Delta_r = \underbrace{\Delta_0 \times \cdots \times \Delta_0}_{r \text{ times}}$$

with the diagonal action of G_0, $\nu = \nu_0 \oplus \cdots \oplus \nu_0 \in H^2(G_0, \Delta_r)$ and the arising extension

$$1 \to \Delta_r \to \Gamma_r \to G_0 \to 1.$$

It is easy to verify that Γ_r is the deck group of a closed flat manifold $d_r(M_0)$ and there is a homomorphism

$$\mathcal{P} : \Gamma_r \ni \gamma_r \to \gamma \in \Gamma_0,$$

inducing the identity map on G_0, such that $(\mathcal{P}|_{\Delta_r})(\delta_1, \dots, \delta_r) = \delta_1$. The holonomy homomorphism φ_r of $d_r(M_0)$ is given by the formula

$$\varphi_r : \Gamma \ni \gamma_r \to \varphi_0(\gamma) \oplus \cdots \oplus \varphi_0(\gamma) \in \mathrm{SO}(n_0) \times \cdots \times \mathrm{SO}(n_0) \subset \mathrm{SO}(rn_0).$$

Proof of Theorem 1.1. By the Auslander–Kuranishi theorem (see [1], [4, ch.3, Theorem 1.1], [18, ch. 3, Theorem 3.4.8]), there is a closed flat manifold V whose holonomy group is isomorphic to G. It is clear that $V_1 = d_2(V)$ is a closed orientable flat manifold. By [7, Theorem 1], $V_2 = d_2(V_1)$ has a spin structure $\widehat{\varphi}_{V_2}$. Let $r = 2|G|$, $M(G) = d_r(V_2)$, $n_2 = \dim V_2$, and $n = \dim M(G)$. The homomorphism

$$\widehat{\varphi} : \Gamma_r \ni \gamma_r \to (\widehat{\varphi}_{V_2}(\gamma), \dots, \widehat{\varphi}_{V_2}(\gamma)) \in \mathrm{Spin}(n_2) \times \cdots \times \mathrm{Spin}(n_2) \subset \mathrm{Spin}(n)$$

determines a spin structure $\mathfrak{s}$ on $M(G)$. From the above description of Σ_n and Σ_{n_2} it follows that

$$\Sigma_n = \underbrace{\Sigma_{n_2} \otimes \cdots \otimes \Sigma_{n_2}}_{r \text{ times}}$$

and the action of $\operatorname{im}\widehat{\varphi}$ on Σ_n is given by the formula

$$\widehat{\varphi}(\gamma_r)(u_1 \otimes \cdots \otimes u_r) = \widehat{\varphi}_{V_2}(\gamma)u_1 \otimes \cdots \otimes \widehat{\varphi}_{V_2}(\gamma)u_r.$$

For every $v \in \Sigma_{n_2}$ consider $\Delta(v) = v \otimes \cdots \otimes v \in \Sigma_n$. Let $q = \dim \Sigma_{n_2}$ and

$$\Delta = \{\Delta(v) \in \Sigma_n : v \in \Sigma_{n_2}\}.$$

Given $g \in \operatorname{im}\widehat{\varphi}$, take a basis $v_1, \ldots, v_q$ of Σ_{n_2} such that $gv_j = z_j v_j$ for $j = 1, \ldots, q$, and for some $z_j \in \mathbb{C}$. The equality $g^r = 1$ implies that $z_j^r = 1$ and consequently

$$g\Delta(v_j) = z_j^r \Delta(v_j) = \Delta(v_j).$$

Since $\Delta(v_1), \ldots, \Delta(v_q)$ is a basis of Δ, $\Delta^g = \Delta$ and $\Delta^{\operatorname{im}\widehat{\varphi}} = \Delta$. This finishes the proof of Theorem 1.1. □

3. Parallel spinors on 4-manifolds

Proof of Theorem 1.2. (a) $\Rightarrow$ (b). Assume that $M = \mathbb{R}^4/\Gamma$ is a closed flat 4-manifold admitting nontrivial parallel spinors. Let A be an element of $\varphi(\Gamma)$ of prime order p, $M_A = \mathbb{R}^4/\varphi^{-1}(\langle A \rangle)$, and let $\mathfrak{s}_A$ be the spin structure on M_A induced by the spin structure $\mathfrak{s}$ on M. Then $\operatorname{Hol}(M_A) = \langle A \rangle$ and $\mathfrak{p}(M_A, \mathfrak{s}_A) \geq \mathfrak{p}(M, \mathfrak{s}) > 0$. Fix $\gamma \in \varphi^{-1}(A)$. The deck transformation γ^p is a translation by some $v \in \mathbb{R}^4 - \{0\}$. As $Av = v$, there is an orthonormal basis e_1, e_2, e_3, e_4 such that $A(e_3) = e_3$, $A(e_4) = e_4$ and A acts as a rotation by an angle ψ on $\operatorname{Span}[e_1, e_2]$. The lift $\widehat{\varphi}(\gamma)$ of $\varphi(\gamma)$ is equal to $\pm(\cos \frac{\psi}{2} + e_1 e_2 \sin \frac{\psi}{2})$.

Since $\dim M = 4$, $p = 2$ or $p = 3$ (see e.g. [5, p. 563]). If $p = 2$, then $\widehat{\varphi}(\gamma)^2 = -1$ and thus $\mathfrak{p}(M_A, \mathfrak{s}_A) = 0$. If $p = 3$, $\psi = \frac{2\pi l}{3}$ for some $l \in \{1, 2\}$ and the action of $\widehat{\varphi}(\gamma)$ on Σ_4 is given by the formula

$$\widehat{\varphi}(\gamma)(u_1 \otimes u_2) = \pm\left(\cos \frac{\pi l}{3} + e_1 e_2 \sin \frac{\pi l}{3}\right)(u_1) \otimes u_2.$$

Since

$$\cos \frac{\pi l}{3} + g_1 g_2 \sin \frac{\pi l}{3} = \begin{bmatrix} \cos \frac{\pi l}{3} & -\sin \frac{\pi l}{3} \\ \sin \frac{\pi l}{3} & \cos \frac{\pi l}{3} \end{bmatrix},$$

$\Sigma_4^{\operatorname{im}\widehat{\varphi}} = \{0\}$. This shows that M has trivial holonomy so that M is a torus. The implication (b) $\Rightarrow$ (a) is obvious. □

Example 3.1. Theorem 1.2 is false for flat manifolds of dimension greater than 4. To see this consider $n \geq 5$, $B = A_3 \oplus A_3 \oplus id_R$, (see Section 5 for the definition of A_3), the translation τ by $(0, 0, 0, 0, \frac{1}{3})$, $\tilde{g} = \tau \circ B$, and the isometry g of T^5 induced by $\tilde{g}$. As g acts freely on T^5, $M = (T^5/\langle g \rangle) \times T^{n-5}$ is a closed flat n-manifold. Take $\alpha_+ \in \mathrm{Spin}(n)$ determined by A. By Theorem 1.6, $\mathfrak{p}(M, \alpha_+) > 0$. Another family of counterexamples form hol$(\mathbb{Z}_p)$-manifolds for $p \geq 7$ (cf. Theorem 1.6). They exist in dimensions not smaller than p ([5, p. 563]).

Remark 3.2.

(a) It is known that every closed flat manifold M, admitting nontrivial parallel spinors, such that $\dim M \leq 3$, is a torus (see e.g. [15, Theorem 5.1]). This result follows easily from our argument.
(b) The problem which higher-dimensional flat manifolds admit nontrivial parallel spinors is much more difficult than the problem considered here. The number $\nu_f(n)$ of n-dimensional closed flat manifolds increases rapidly with n. It is known that $\nu_f(2) = 2$, $\nu_f(3) = 10$ ([18, Section 3.5]), $\nu_f(4) = 74$, $\nu_f(5) = 1060$ ([5]), and $\nu_f(6) = 38746$ ([5]). For more information about affine equivalence classes of closed flat manifolds up to dimension 6 we refer to Carat home page at http://wwwb.math.rwth-aachen.de/carat/.

4. Parallel spinors on flat manifolds with cyclic holonomy groups

Throughout this section $M = \mathbb{R}^n/\Gamma$ is a closed, orientable, flat manifold with holonomy group $\varphi(\Gamma)$ isomorphic to $\mathbb{Z}_r$. Define an endomorphism ρ of $\mathbb{C}^2$ by the formula

$$\rho(u) = \cos\beta\, u + \sin\beta\, g_1 g_2\, u.$$

For $j \in \mathbb{Z}$ the matrix of ρ^j is equal to

$$\cos(\beta j)\, I + \sin(\beta j) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos(\beta j) & -\sin(\beta j) \\ \sin(\beta j) & \cos(\beta j) \end{bmatrix}.$$

In the proofs of Theorems 1.3 and 1.4 we use the following lemma (compare [16, Lemma 7]).

Lemma 4.1. *Let $w_{+1} = (1, -i)$, $w_{-1} = (1, i)$, $\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \{-1, 1\}^k$, and $v_\epsilon = w_{\epsilon_1} \otimes \cdots \otimes w_{\epsilon_k}$. Take $\beta = \frac{\pi}{r}$ and μ_ϵ as above. Let $u = u_1 \otimes \cdots \otimes u_k \in \Sigma_n$. Then*

(a) $\alpha u = \rho^{C_1}(u_1) \otimes \cdots \otimes \rho^{C_k}(u_k)$,
(b) $\rho(w_{\pm 1}) = e^{\pm i\beta} w_{\pm 1}$,
(c) $\alpha v_\epsilon = e^{i\beta\mu_\epsilon} v_\epsilon$ *and* $\{v_\epsilon : \epsilon \in \{-1, 1\}^k\}$ *is a basis of* Σ_n.

Proof.

(a) Since $T^2 = I$,

$$e_{2j-1}e_{2j}(u_1 \otimes \cdots \otimes u_j \otimes \cdots \otimes u_k) = u_1 \otimes \cdots \otimes (g_1 g_2)(u_j) \otimes \cdots \otimes u_k$$

and

$$\begin{aligned}&((\cos\,\psi_1 + e_{2j-1}e_{2j}\,\sin\,\psi_1)(\cos\,\psi_2 + e_{2j-1}e_{2j}\,\sin\,\psi_2))\\ &\quad(u_1 \otimes \cdots \otimes u_j \otimes \cdots \otimes u_k) = u_1 \otimes \cdots \otimes (\cos(\psi_1+\psi_2)\\ &\qquad + e_{2j-1}e_{2j}\,\sin(\psi_1+\psi_2))(u_j) \otimes \cdots \otimes u_k.\end{aligned} \tag{1}$$

The equality $\beta_j = C_j\beta$ implies that

$$\alpha(u_1 \otimes \cdots \otimes u_k) = (\rho_1, \ldots, \rho_k)(u_1 \otimes \cdots \otimes u_k) = \rho^{C_1}(u_1) \otimes \cdots \otimes \rho^{C_k}(u_k).$$

(b) A direct calculation yields

$$\rho(w_{+1}) = \rho(1, -i) = e^{i\beta} w_{+1}$$

and

$$\rho(w_{-1}) = e^{-i\beta} w_{-1}.$$

(c) By (b), $\rho(w_{\epsilon_j}) = e^{i\beta\epsilon_j} w_{\epsilon_j}$. Hence

$$\alpha(v_\epsilon) = \rho^{C_1}(w_{\epsilon_1}) \otimes \cdots \otimes \rho^{C_k}(w_{\epsilon_k}) = e^{i\beta\mu_\epsilon} v_\epsilon.$$

Since $\#\{v_\epsilon : \epsilon \in \{-1, 1\}^k\} = 2^k = \dim \Sigma_n$ and the vectors v_ϵ are linearly independent, they form a basis of Σ_n. This finishes the proof of Lemma 4.1. □

Proof of Theorem 1.3. We have

$$\alpha^r v_\epsilon = e^{i\pi \sum_{j=1}^{k} C_j} v_\epsilon = \begin{cases} v_\epsilon & \text{if } \mu_I \text{ is even} \\ -v_\epsilon & \text{if } \mu_I \text{ is odd} \end{cases}$$

Since r is odd, $(-\alpha)^r = -\alpha^r$ so that $\alpha_+^r = 1$ and $\alpha_-^r = -1$. In particular, there is exactly one admissible flat spin structure on M, namely the spin structure determined by α_+.

If μ_I is even, then $\alpha_+ = \alpha$ and

$$\alpha_+ v_\epsilon = v_\epsilon \Leftrightarrow e^{i\beta\mu_\epsilon} v_\epsilon = v_\epsilon \Leftrightarrow \mu_\epsilon \equiv 0 \bmod (2r).$$

If μ_I is odd, then $\alpha_+ = -\alpha$ and

$$\alpha_+ v_\epsilon = v_\epsilon \Leftrightarrow -e^{i\beta\mu_\epsilon} v_\epsilon = v_\epsilon \Leftrightarrow \mu_\epsilon \equiv r \bmod (2r).$$

This completes the proof of Theorem 1.3. □

Proof of Theorem 1.4. We have

$$(\pm\alpha)^r v_\epsilon = \alpha^r v_\epsilon = e^{i\mu_I\pi} v_\epsilon = (-1)^{\mu_I} v_\epsilon.$$

If μ_I is odd, then $(\pm\alpha)^r = -1$ so that there are no admissible spin structures on M. If μ_I is even, then $(\pm\alpha)^r = 1$,

$$\alpha v_\epsilon = v_\epsilon \Leftrightarrow \mu_\epsilon \equiv 0 \bmod (2r), \quad \text{and} \quad (-\alpha) v_\epsilon = v_\epsilon \Leftrightarrow \mu_\epsilon \equiv r \bmod (2r).$$

The proof of Theorem 1.4 is complete. □

Proof of Corollary 1.5. Let $\mathcal{D}_\pm = \{\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \{-1, 1\}^k : \epsilon_1 = \pm 1\}$ and $l \in \{0, 1\}$. The map $f(\epsilon) = -\epsilon$ is a bijection carrying $\mathcal{D}_+$ onto $\mathcal{D}_-$ and $\{-1, 1\}^k = \mathcal{D}_+ \cup \mathcal{D}_-$. Using the relation

$$\mu_\epsilon \equiv lr \bmod (2r) \Leftrightarrow \mu_{f(\epsilon)} \equiv lr \bmod (2r)$$

we conclude the proof of Corollary 1.5. □

5. Parallel spinors on hol($\mathbb{Z}_p$)-manifolds

Let p be a prime number. The aim of this section is to prove Theorems 1.6 and 1.7. Fix a generator A of $\varphi(\Gamma)$. The group ring $\mathbb{Z}[\mathbb{Z}_p]$ consists of formal sums $\sum_{j=0}^{p-1} n_j g^j$ such that g is a generator of $\mathbb{Z}_p$ and $n_j \in \mathbb{Z}$. The element $\Sigma = \sum_{j=0}^{p-1} g^j$ generates an ideal $I(\Sigma)$ of $\mathbb{Z}[\mathbb{Z}_p]$ and $\mathbb{Z}[\zeta_p] = \mathbb{Z}[\mathbb{Z}_p]/I(\Sigma)$. The isomorphism

$$\widehat{A}_p : \mathbb{R}^p \ni (x_1, x_2, \ldots, x_p) \to (x_2, \ldots, x_p, x_1) \in \mathbb{R}^p$$

acts trivially on $W = \{(x_1, \ldots, x_1) : x_1 \in \mathbb{R}\}$ so that $\widehat{A}_p$ induces $A_p : \mathbb{R}^p/W \to \mathbb{R}^p/W$. We can identify $\varphi(\Gamma)$ with $\mathbb{Z}_p$, $\mathbb{R}^p/W$ with $\mathbb{R}^{p-1}$, and $\widehat{A}_p$ with $A_p \oplus id_R$.

Lemma 5.1.

(*a*) *A is conjugate to the direct sum of $l(M)$ copies of A_p and $id_{R^{n-(p-1)l(M)}}$.*
(*b*) *If $p > 2$*, then *A_p is the direct sum of rotations by $\frac{2\pi}{p} j$, $j = 1, \ldots, \frac{p-1}{2}$.*

Proof.

(a) Since A has integral coefficients, it induces the structure of a $\mathbb{Z}[\mathbb{Z}_p]$-module ξ_A in $\mathbb{Z}^n$. It is clear that the $\mathbb{Z}[\mathbb{Z}_p]$-modules corresponding to $\widehat{A}_p$ and A_p are $\mathbb{Z}[\mathbb{Z}_p]$ and $\mathbb{Z}[\zeta_p]$. Let $\mathbb{Z}_{(p)}$ denote the p-localization of $\mathbb{Z}$. It is known that every nontrivial irreducible $\mathbb{Z}[\mathbb{Z}_p]$-module after a p-localization is isomorphic to $\mathbb{Z}[\mathbb{Z}_p] \otimes \mathbb{Z}_{(p)}$ or $\mathbb{Z}[\zeta_p] \otimes \mathbb{Z}_{(p)}$ (see e.g. [5, p. 553]). Moreover, $\mathbb{Z}[\mathbb{Z}_p]^{\mathbb{Z}_p} \cong \mathbb{Z}$ and $\mathbb{Z}[\zeta_p]^{\mathbb{Z}_p} \cong \{0\}$. Thus there is an integer l such that $\xi_A \otimes \mathbb{R}$ is isomorphic to the direct sum of l copies of $\mathbb{Z}[\zeta_p] \otimes \mathbb{R}$ and $(\xi_A \otimes \mathbb{R})^{Z_p} \cong \mathbb{R}^{n-(p-1)l}$. The last claim implies that $l = l(M)$.

(b) Applying the Laplace expansion with respect to the last column of the matrix of $\widehat{A}_p - z\,id$ it is easy to see that the characteristic polynomial of $\widehat{A}_p$ is equal to $(-1)^p z^p + (-1)^{p+1}$ and the eigenvalues of $\widehat{A}_p$ are equal to $e^{\frac{2\pi j}{p}}$ for $j = 0, \dots, p-1$. □

Let $l = l(M)$, $k = [n/2]$, $k(2) = 1$, and $k(p) = \frac{p-1}{2}$ for $p > 2$. Every element ϵ of $\{-1, 1\}^k$ can be written as

$$\epsilon = (\epsilon(1), \dots, \epsilon(l), \widehat{\epsilon})$$

for some $\epsilon(i) = (\epsilon_{i,1}, \dots, \epsilon_{i,k(p)}) \in \{-1, 1\}^{k(p)}$, $i = 1, \dots, l$, $\widehat{\epsilon} \in \{-1, 1\}^{k-k(p)l}$. Let

$$\mu_{\epsilon(i)} = \sum_{j=1}^{k(p)} j\epsilon_{i,j} \quad \text{and} \quad \mu(p) = \frac{k(p)(k(p)+1)}{2}.$$

By Lemma 5.1,

$$\mu_\epsilon = \sum_{i=1}^{l} \mu_{\epsilon(i)} \quad \text{and} \quad \mu_I = l\mu(p).$$

We use the following observation proved in [16, Section 4].

Lemma 5.2. *If $p > 5$, then there is $\epsilon^* \in \{-1, 1\}^{k(p)}$ such that*

$$\mu_{\epsilon^*} = \begin{cases} 0 & if\ \mu(p)\ is\ even \\ p & if\ \mu(p)\ is\ odd \end{cases}.$$

Proof of Theorem 1.6. First consider the case $p > 5$. Let $\epsilon^*, \widehat{\epsilon}$ be as above and let

$$\epsilon = (\underbrace{\epsilon^*, \dots, \epsilon^*}_{l\text{ copies}}, \widehat{\epsilon}).$$

Then $\mu_\epsilon = l\mu_{\epsilon^*} \equiv 0 \bmod (2p)$, if $\mu_I = l\mu(p)$ is even, and $\mu_\epsilon = lp \equiv p \bmod (2p)$, if μ_I is odd. Assume that $p = 3$ and $l \geq 2$. Then $l = 2l_1$ or $l = 2l_2 + 3$. Take $\delta = (1, -1)$, $\epsilon = (\underbrace{\delta, \dots, \delta}_{l_1\text{ copies}}, \widehat{\epsilon})$ in the first case and $\epsilon' = (1, 1, 1, \underbrace{\delta, \dots, \delta}_{l_2\text{ copies}}, \widehat{\epsilon})$ in the second. It is obvious that $\mu_\epsilon = 0$ and $\mu_{\epsilon'} = 3$.

Consider the next case $p = 5$ and $l \geq 2$. As above, $l = 2l_1$ or $l = 2l_2 + 3$. Let $\delta \in \{-1, 1\}^2$ and $\omega = (1, 1, 1, 1, 1, -1)$. Take $\epsilon = (\delta, -\delta, \dots, \delta, -\delta, \widehat{\epsilon})$ in the first case and $\epsilon' = (\omega, \delta, -\delta, \dots, \delta, -\delta, \widehat{\epsilon})$ in the second. Then $\mu_\epsilon = 0$ and $\mu_{\epsilon'} = \mu_\omega = 5$.

It is easily seen that the equation $\mu_\epsilon \equiv p \bmod (2p)$ has no solutions for $p \in \{3, 5\}$ and $l = 1$. This finishes the proof of Theorem 1.6. $\square$

Proof of Theorem 1.7. A generator A of $\varphi(\Gamma)$ is the direct sum of say l copies of the rotation by π. Since $(\mathbb{R}^n)^A \cong \mathbb{R}^{n-2l}$, $l = l(M)$. By Theorem 1.4, $\mathfrak{p}(M, \alpha) = \mathfrak{p}(M, -\alpha) = 0$ for l odd. Assuming that $l = 2l_1$ consider

$$\epsilon = (1, -1, \dots, 1, -1, \widehat{\epsilon}) \qquad \text{and} \qquad \epsilon' = (1, 1, 1, -1, \dots, 1, -1, \widehat{\epsilon}).$$

Then $\mu_\epsilon = 0$ and $\mu_{\epsilon'} = 2$. The first equality implies that $\mathfrak{p}(M, \alpha) > 0$ and the second that $\mathfrak{p}(M, -\alpha) > 0$. This completes the proof of Theorem 1.7. $\square$

Remark 5.3. The arguments given in the proofs of Theorems 1.6 and 1.7 can be used to estimate $\mathfrak{p}(M, \mathfrak{s})$ from below. For simplicity consider only the case $p > 7$. Let $c(p) = \left[\frac{k(p)}{4}\right]$. The sequence δ_4 considered in the proof of Theorem 1 in [14] can be replaced by $-\delta_4$ so that the number of ϵ^* from Lemma 5.2 is greater or equal $2^{c(p)+1}$ (cf. [16, Section 4]) and thus

$$\mathfrak{p}(M, \alpha_+) \geq 2^{(c(p)+1)l(M)+k-k(p)l(M)}.$$

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